

# Equity Objectives in Vehicle Routing: A Survey and Analysis

**P. Matl\***

University of Vienna, Austria  
piotr.matl@univie.ac.at

**R.F. Hartl**

University of Vienna, Austria  
richard.hartl@univie.ac.at

**T. Vidal**

Pontifical Catholic University of Rio de Janeiro, Brazil  
vidalt@inf.puc-rio.br

Over the past two decades, equity aspects have been considered in a growing number of models and methods for vehicle routing problems (VRPs). Equity concerns most often relate to fairly allocating workloads and to balancing the utilization of resources, and many practical applications have been reported in the literature. However, there has been only limited discussion about how equity should be modelled in the context of VRPs, and various measures for optimizing such objectives have been proposed and implemented without critical evaluation of their respective merits and consequences.

This article addresses this gap by providing an analysis of classical and alternative equity measures for bi-objective VRP models. In the survey we review and categorize the existing literature on equitable VRP models. In the analysis, we identify a set of axiomatic properties which an ideal equity measure should satisfy, collect six common measures of equity, and point out important connections between their properties and the properties of the resulting Pareto-optimal solutions. To gauge the extent of some of these implications and to examine further relevant aspects for choosing an equity objective, we conduct also a numerical study on small bi-objective VRP instances solvable to optimality. We find that the equity of cost-optimal solutions is generally quite poor, while the marginal cost of improvement is very low for all examined measures. We also reveal two contradictions of optimizing equity with non-monotonic measures: Pareto-optimal solutions can consist of non-TSP-optimal tours, and even if all tours are TSP-optimal, Pareto-optimal solutions can be composed of tours which are all equal to or longer than those of other Pareto-optimal solutions. Based on these analyses, we conclude that monotonic equity measures are more appropriate for certain types of VRP models, and suggest several promising avenues for further research on equity in logistics applications.

**Keywords:** multi-objective vehicle routing, equity measures, balancing, workload distribution, logistics.

\* Corresponding author

# 1 Introduction

Logistics is multi-objective – the complexity of real-life logistics planning and decision-making often cannot be reduced to cost only. From a more holistic point of view, non-monetary factors such as service quality, reliability, customer satisfaction, and consistency can be decisive for an organization’s medium or long-term performance, especially if those factors can be improved at little additional cost (Jozefowicz et al. 2008, Kovacs et al. 2014).

As a consequence, concerns about various forms of equity have gained increasing attention during the past two decades. This is not surprising, since the explicit consideration of equity issues can provide tangible benefits to organizations. For example, balancing the capacity utilization of vehicles can free up bottleneck resources for future growth in operations without the need for additional investment (Apte and Mason, 2006). Equitable workload allocation can also improve the acceptance of operational plans among drivers, their morale, and the quality of customer service provided (Liu et al., 2006). The relevance of these concerns is reflected in the number of practical applications found in the operations research and logistics literature (Karsu and Morton, 2015).

Despite this interest in equity, there has been only limited discussion about how to model equity concerns in the context of vehicle routing problems. A variety of equity objectives have been proposed and implemented, but the consequences of any specific equity objective choice remain largely unclear. This important modelling choice is often taken as given, and observations reported thus far in the literature are mostly of an empirical nature. As a result, there is a lack of general guidelines for which *types* of measures are appropriate in which contexts. Yet such guidance is essential given the broad variety of surveyed applications. Furthermore, some paradoxes such as “artificial equity” have been observed empirically (Jozefowicz et al., 2002; Halvorsen-Weare and Savelsbergh, 2016), but remain unexplained in the general case.

Although single-objective comparative analyses of some equity objectives have been conducted (Campbell et al., 2008; Huang et al., 2012; Bertazzi et al., 2015), the conclusions of these studies are of limited scope in more general multi-objective settings. Indeed, it is not obvious to what extent single-objective worst-case comparisons reflect the properties of intermediate trade-off solutions. Yet in practical applications, the primary interest lies ultimately in compromise solutions rather than any single-objective optimum.

In this survey and analysis, we take a step back and re-evaluate different ways of accommodating equity considerations in multi-objective VRP models and methods. Our contributions are fourfold:

- we review and categorize the existing literature on models and methods for equitable VRPs (Section 2),
- we analyse the choice of equity measure from an axiomatic perspective and compare six common measures (Sections 3.1 to 3.3),
- we outline relevant connections between the properties of the chosen equity measure and the properties of Pareto-optimal solution sets (Section 3.4), and
- we conduct a numerical study to estimate the extent of some of these connections and to provide further insight into other relevant aspects for choosing an equity measure (Section 4).

We conclude this article in Section 5 by summarizing our findings and suggesting a number of relevant directions for research on equity aspects in logistics applications.

## 2 Models for Equity in Vehicle Routing

Table 1 classifies the current literature on vehicle routing with equity considerations. Based on our search, this literature can be divided into five groups. A first stream of research focuses on a prototypical bi-objective CVRP model with equity as the second objective (called the vehicle routing problem with route balancing – VRPRB), and a second growing group of papers proposes time window extensions to this basic model. A third set of publications deal with the single-objective “min-max VRP”, in which equity is the primary optimization objective. Some other works focus on incorporating equity issues into more complex VRP variants such as periodic VRPs or arc routing problems. These publications build the fourth stream. Finally, a large and varied selection of VRP application papers rounds out the fifth group.

In this article we make a distinction between equity *metrics*, equity *functions*, and equity *objectives*. With equity metrics we refer to the quantity that is balanced, e.g. work-load can be measured by the tour length, the demand served, etc. It is useful to distinguish between metrics whose sum is constant over all VRP solutions (e.g. demand in typical

Publication (by year)	Equity Function			Equity Metric		Optimization Model			
	range	min-max	other	length	demand	MO	WS	CN	PO
<b>Standard VRPRB</b>									
Lacomme et al. (2015)	×			×		×			
Oyola, Løkketangen (2014)	×			×		×			
Sarpong et al. (2013)		×		×	×	×			
Reiter, Gutjahr (2012)		×		×		×			
Jozefowiez et al. (2009)	×			×		×			
Borgulya (2008)	×			×		×			
Jozefowiez et al. (2007)	×			×		×			
Pasia et al. (2007a)	×			×		×			
Pasia et al. (2007b)	×			×		×			
Jozefowiez et al. (2006)	×			×		×			
Jozefowiez et al. (2002)	×			×		×			
<b>VRPRB with Time Windows</b>									
de Freitas Aquino, Arroyo (2014)	×			×		×			
Melián-Batista et al. (2014)	×			×		×			
Baños et al. (2013)	×			×	×	×			
<b>Min-Max VRP</b>									
Bertazzi et al. (2015)	×	×		×					×
Wang et al. (2014)		×		×					×
Narasimha et al. (2013)		×		×					×
Schwarze, Voß (2013)		×		×					×
Yakıcı, Karasakal (2013)		×		×					×
Huang et al. (2012)	×	×	×	×	×		×		
Valle et al. (2011)		×		×					×
Carlsson et al. (2009)		×		×					×
Campbell et al. (2008)	×	×		×					×
Saliba (2006)		×		×					×
Applegate et al. (2002)		×		×					×
Golden et al. (1997)		×		×					×
<b>Other VRPs with Equity Aspects</b>									
Halvorsen-Weare, Savelsbergh (2015)	×	×	×	×		×			
Mandal et al. (2015)	×			×		×			
Martínez-Salazar et al. (2014)	×			×		×			
Liu et al. (2013)		×		×					×
Gulczynski et al. (2011)	×				×		×		
Mei et al. (2011)		×		×		×			
Mourgaya, Vanderbeck (2007)		×			×			×	
Lacomme et al. (2006)		×		×		×			
Ribeiro, Lourenço (2001)			×		×		×		
<b>Application Papers</b>									
de Armas et al. (2015)	×			×				×	
Goodson (2014)	×			×				×	
López-Sánchez et al. (2014)		×		×					×
Bektaş (2013)	×			×				×	
Rienthong et al. (2011)	×			×					×
Kritikos, Ioannou (2010)			×		×		×		
Groër et al. (2009)			×	×	×			×	
Mendoza et al. (2009)			×		×			×	
Apte, Mason (2006)	×			×	×			×	
Jang et al. (2006)			×	×			×		
Kim et al. (2006)	×			×				×	
Lin, Kwok (2006)	×			×	×	×			
Liu et al. (2006)	×			×	×			×	
Pacheco, Martí (2006)		×		×		×			
Blakeley et al. (2003)			×		×		×	×	
Corberán et al. (2002)		×		×		×			
Serna, Bonrostro (2001)		×		×					×
Lee, Ueng (1999)			×	×			×		
Bowerman et al. (1995)			×		×		×		

Table 1: Vehicle Routing Publications Incorporating Equity Concerns

VRPs), and those for which the sum is variable (e.g. tour length). An equity function specifies how an index value is calculated for a given allocation of workloads, e.g. using the range or standard deviation. We use the terms equity function and equity measure interchangeably. Finally, an equity objective refers to a specific combination of metric and function, e.g. the equity objective in the VRPRB is the range of tour lengths.

Looking at the identified literature as a whole, the majority of authors use tour length as the equity metric, based either on distance or duration. Fewer papers consider workload in terms of the demand served per tour. The two most common functions used to measure equity at the solution level are the largest workload (min-max), and the difference between the largest and the smallest (the range). Other functions, such as the standard deviation or the average deviation to the mean, are used in a few applications.

Various optimization models are used to tackle the equity objective. In single-objective models, equity is either the primary objective (PO), or it is modelled as a constraint (CN). In bi-objective models, equity is treated as a separate objective and the set of efficient solutions is generated either with a weighted sum approach (WS) or with multi-objective methods (MO). In the following, we review in more detail the research in each of the identified streams.

## 2.1 The VRP with Route Balancing

Among the papers surveyed, one standard bi-objective problem formulation has gained traction in the “theoretical” literature, namely the VRPRB. This model extends the basic capacitated VRP (CVRP) with a second objective for equity, using the range measure based on distance travelled.

The VRPRB was first introduced in Jozefowiez et al. (2002), along with a parallel genetic algorithm with tabu search. Further work by the same authors was published in Jozefowiez et al. (2006) with adaptations to NSGA-II based on parallel search and enhanced management of population diversity. This was followed by a paper introducing target-aiming Pareto search, a hybridization of NSGA-II, tabu search, and goal programming techniques (Jozefowiez et al. 2007).

A number of competing solution procedures have since been proposed by other authors. Pasia et al. (2007a) combine a randomized Clarke-Wright savings algorithm with Pareto local search. A comparison with the results of Jozefowiez et al. (2006) suggests that their procedure generates better efficient sets in terms of the hypervolume, unary epsilon, and R3 metrics. This work is further extended in Pasia et al. (2007b) with the use of Pareto ant colony optimization, yielding improved results. Borgulya (2008) proposes an evolutionary

algorithm in which recombination is omitted in favour of an adaptive mutation operator. Finally, Jozefowiez et al. (2009) present a classical evolutionary algorithm extended with parallel search and elitist diversification management, finding that both new mechanisms contribute noticeably to improving the quality of the generated non-dominated solutions.

In contrast to earlier approaches, more recent contributions do not apply evolutionary algorithms. Oyola and Løkketangen (2014) present a GRASP heuristic combined with local search. On small instances, their approach matches or outperforms the best bounds found by an exact method based on a weighted sum objective. The authors also reimplemented a sequential version of the algorithm proposed in Jozefowiez et al. (2009), yielding slightly better solution quality, but at the expense of considerably higher computation times. The most recent article on the VRPRB (Lacomme et al., 2015) introduces a multi-start decoder-based method which alternates between direct and indirect solution spaces. Non-dominated sets of solutions are extracted from TSP giant tour representations through a bi-objective split procedure. The resulting VRPRB solutions serve as starting points for further search based on path re-linking, and a multi-start strategy is applied for additional diversification.

To the best of our knowledge, Reiter and Gutjahr (2012) present the only exact algorithm designed specifically for a bi-objective VRP with an equity objective. However, in this work equity is measured with the maximum tour length, rather than the range of lengths adopted in the VRPRB. The authors propose an adaptive  $\epsilon$ -constraint method based on a hybridization of branch-and-cut and two genetic algorithms. An interesting conclusion of their study is that computing higher equity (or higher cost) solutions requires considerably more computational effort than for lower equity (or lower cost) solutions. A more general model is investigated in Sarpong et al. (2013). The authors consider bi-objective VRPs with general min-max objectives, and a column generation algorithm is proposed to solve these problems to optimality.

## 2.2 Time Window Extensions to the VRPRB

Using the VRPRB model as a starting point, several publications consider the addition of time windows to the VRPRB. Baños et al. (2013) examine the VRPTW with a range equity objective based on either tour lengths or tour loads. The authors propose an evolutionary algorithm with a bi-objective acceptance criterion based on simulated annealing, and show that the approach outperforms NSGA-II and SPEA2 in computational experiments. A comparison of the fronts reported in Baños et al. (2013) suggests that the length and load based equity objectives can lead to noticeable differences in the solutions.

A hybrid heuristic combining iterated local search, randomized variable neighbourhood descent, and genetic recombination is proposed in de Freitas Aquino and Arroyo (2014). The authors compare their approach with NSGA-II and a reimplement of the algorithm in Baños et al. (2013), finding that their approach outperforms the others.

A multi-objective scatter search heuristic for the VRPRB with time windows is presented in Melián-Batista et al. (2014). The authors show that the method approximates well the exact Pareto front on small instances, and outperforms NSGA-II on larger ones. The authors note that the addition of time windows introduces a problem when equity is optimized: unnecessary waiting times may be added to tours in order to artificially improve the equity objective.

## 2.3 The Min-Max VRP

In contrast to the provider-oriented perspective of minimizing total cost, service and client-oriented logistics applications often consider equity and balance issues as the primary optimization objective. For example, this is the case in school bus routing (minimizing the longest time spent in the vehicle in Serna and Bonrosto 2001), delivery of newspapers or other perishable goods (minimizing the latest delivery time in Applegate et al. 2002), communication or computer networks (minimizing the maximum latency in Nace and Pióro 2008), humanitarian relief (minimizing the latest arrival of aid in Campbell et al. 2008), as well as military reconnaissance or surveillance operations (minimizing the longest mission time in Carlsson et al. 2013). A common feature of many of these applications is that the equity objective is defined with the min-max measure, and post-optimization procedures are applied to optimize the remaining parts of the solution if necessary.

An early contribution to the class of min-max VRP problems can be found in Golden et al. (1997). The authors apply the min-max objective to the CVRP, the CVRP with multiple use of vehicles, and the m-TSP, proposing a tabu-search-based adaptive memory heuristic as a solution procedure. Their computational experiments show that the total cost of the best min-max solution can be significantly higher than the cost of the best min-sum solution. Applegate et al. (2002) revisit a min-max VRP instance from a competition based on newspaper delivery. The authors employ linear programming (LP) and cutting planes within a distributed branch-and-bound framework to prove the optimality of the heuristic solution which had won the original competition. Saliba (2006) presents construction heuristics for a lexicographic version of the min-max VRP.

Extensions to the min-max VRP have recently gained attention. The multi-depot variant (min-max MDVRP) was first introduced in Carlsson et al. (2009). Their paper

presents a theoretical analysis of optimal solutions and compares three different heuristic solution approaches, including LP-based and region partitioning methods. This work is further pursued in Narasimha et al. (2013), in which an ant colony optimization method is combined with equitable region partitioning to solve a series of single-depot problems. Wang et al. (2014) also extend some of the ideas presented in Carlsson et al. (2009) to the min-max MDVRP and compare the performance of four different heuristics.

More complex extensions of the single-depot case are presented in Yakıcı and Karasakal (2013) and Schwarze and Voß (2013). Yakıcı and Karasakal (2013) consider a min-max VRP with multiple regions, multiple types of customer demand, and a heterogeneous fleet of vehicles, with the possibility of split deliveries. A multi-stage heuristic procedure is presented to solve the problem. Schwarze and Voß (2013) reconsider the Skill VRP, in which demands have different levels of skill requirements that must be met by the serving vehicles. Based on the observation that solutions to the standard model produce TSP-like solutions with poor capacity utilization, an alternative model with a min-max objective is introduced to better balance the utilization of available resources.

Finally, Valle et al. (2011) describe the min-max selective VRP, a variant encountered in designing wireless sensor networks. In this problem, it is not required to visit all the nodes, but those not visited must be within some distance of a visited node. As in other min-max VRPs, the objective is to minimize the length of the longest route. The authors propose two exact approaches as well as a hybrid metaheuristic to solve this problem.

Since all of the above publications propose single-objective models, an important question is to what extent the chosen objective affects the properties and structure of optimal solutions. This topic is studied in Campbell et al. (2008); Huang et al. (2012), and Bertazzi et al. (2015). These articles will be reviewed in Section 3.

## 2.4 Other VRPs with Equity Aspects

Some studies examine equity objectives from a multi-period perspective. Gulczynski et al. (2011) consider two variants of the periodic VRP (PVRP), including one with an equity objective. An interesting aspect of their model is that the workload is measured by the number of customers per tour, which can be the main determinant of workload in certain applications. Gulczynski et al. (2011) measure equity with the range of workloads on a given day and combine this with the classical total cost objective into a weighted sum objective function. The problem is solved with an integer programming based heuristic.

Mourgaya and Vanderbeck (2007) examine the PVRP from a tactical perspective, with the aims of regional compactness and workload balance among routes. In this model



the workload is measured by total customer demand. Mourgaya and Vanderbeck (2007) measure equity according to the min-max function and handle it as a constraint in a column generation based heuristic.

In contrast, workload equity is the main optimization objective in the PVRP considered in Liu et al. (2013). The problem is motivated by home healthcare logistics applications, in which reasonable working hours are a major concern. The authors therefore minimize the maximum route duration as the primary objective, and propose a tabu search heuristic for this purpose. Finally, Ribeiro and Lourenço (2001) consider a PVRP with three objectives: minimization of total cost, minimization of the standard deviation of route durations, and minimization of an inconsistency objective for customer service. However, only a small instance is solved due to the non-linearity of the proposed model.

Martínez-Salazar et al. (2014) present a transportation location routing problem motivated by soft drink distribution. The strategic decisions of locating distribution centres are jointly optimized with the operational routing decisions. The authors formulate a bi-objective model that minimizes total system cost and minimizes the imbalance between the lengths of the delivery routes, defined with the range function. A scatter search and NSGA-II, both combined with tabu search, are proposed for solving the problem.

Several publications deal with bi-objective capacitated arc routing problems (CARPs) with equity objectives. Lacomme et al. (2006) consider the trade-off between total distance minimization and minimization of the maximum tour cost, and propose an adaptation of NSGA-II for determining the set of efficient solutions. The bi-objective CARP is examined also in Mei et al. (2011), in which a decomposition-based memetic algorithm is proposed. An extensive comparative study with the algorithm of Lacomme et al. (2006) and a standard NSGA-II shows that the approach described in Mei et al. (2011) yields significant improvements in solution quality, but at the expense of noticeably larger computation times.

A generalization of the CARP and CVRP – the mixed capacitated general routing problem (MCGRP) – is studied in Mandal et al. (2015). The authors measure equity with the range function based on route cost, and propose a memetic version of NSGA-II that uses a Pareto-based local search procedure. Computational experiments suggest that some crossover operators are considerably more effective than others for optimizing the balance criterion. In Halvorsen-Weare and Savelsbergh (2016) small instances of the MCGRP are solved exactly with four different functions for the equity objective. We review the observations and conclusions of this study in more detail in Section 3.

## 2.5 Applications

Some of the earliest VRP applications with equity aspects focus on school bus routing. Since these transportation services are generally provided by the public sector, equity objectives must be considered in addition to cost efficiency. Bowerman et al. (1995) consider the design of school bus routes in an urban setting in Canada. Next to minimizing the number of bus routes, equity objectives are included to balance the route lengths and the number of students served by the routes, as well as to minimize student walking distance. The authors propose a two-phase solution approach based on equitable districting for minimizing the number of routes and balancing their load, followed by a routing algorithm for selecting the bus stop locations to minimize student walking distance. In Serna and Bonrostro (2001), a rural school bus routing problem in Spain is re-examined from a social perspective. A min-max VRP model is optimized with the aim of minimizing the longest time spent in transit. Computational tests show that the min-max solution can significantly improve the service objective compared to the solution of the standard min-sum model, while still reducing costs compared to manual plans.

Corberán et al. (2002) extend the min-max VRP model of Serna and Bonrostro (2001) with an explicit second objective in order to capture the trade-off between operational cost and service quality. The objectives are to minimize the number of bus routes and minimize the maximum travel time. The authors present a scatter search heuristic for determining the efficient set of solutions. The bi-objective problem is further investigated in Pacheco and Martí (2006), who examine the performance of four construction heuristics and propose a tabu search combined with path re-linking to obtain improved solutions compared to those in Corberán et al. (2002). More recently, López-Sánchez et al. (2014) consider the planning of bus services for employees of a large company. The problem is modelled as an open VRP with a min-max objective in order to balance the time spent in transit and comply with insurance constraints. A multi-start heuristic is presented for solving the problem, and also achieves competitive results on previous school bus routing instances.

Equity concerns also appear as objectives in a variety of other applications. Lin and Kwok (2006) examine a location routing problem faced by a telecommunications company in Hong Kong. The authors solve a multi-objective formulation with the aim of minimizing total cost as well as range equity objectives for working hours and for demand served. They also consider how allowing multiple trips per employee affects the optimization of the equity objectives. Two heuristics based on tabu search and simulated annealing are compared. de Armas et al. (2015) present a rich VRP application for a logistics services

provider in Spain. In addition to the classical objectives of minimizing total cost and the number of vehicles, the authors also consider range measures for route duration and route distance. These objectives are optimized in a hierarchical fashion based on user input.

In other applications, equity concerns are handled with constraints. Liu et al. (2006) consider the planning of third party logistics services to convenience stores in Taiwan. In order to reduce overtime and improve acceptance of the routing plan among drivers, the total travel distance is minimized subject to constraints on the duration and the load of the vehicles. Goodson (2014) describes the election day routing of rapid response attorneys to poll observers in the United States. The fair allocation of workload is particularly important in this context because the attorneys are volunteers. Two alternative models are examined, one with equity based on the number of assigned polling stations per attorney, and the other based on equity of route durations between assigned polling stations.

Groër et al. (2009) present a balanced billing cycle VRP encountered by an utility company. When accounts are cancelled or opened, fixed meter reading routes must be adjusted periodically to maintain efficiency while meeting regulatory requirements and customer service considerations. At the same time, it is desirable to balance the workload per day to avoid fixed periodic overtime costs and peaks in administrative work. This is handled by lower and upper bound constraints on the meter route distance and the number of meters serviced per route. A similar application case is described in Blakeley et al. (2003) for a complex technician routing and scheduling problem. The authors optimize periodic maintenance operations of elevators at various customer sites by minimizing a weighted sum of total travel cost, overtime, and workload balance, the latter being based on customer service times. In addition, upper and lower bounds on the workload of each route are included, as it is noted that some weight combinations would otherwise lead to unacceptable day-to-day variations of workload.

Some applications arise also when optimizing delivery operations at libraries. Apte and Mason (2006) investigate the delivery of requested, returned, and new items between the branches of a large urban library network in San Francisco. Because the capacity of a delivery system depends critically on bottleneck resources, balancing the utilization of available truck capacities is essential for handling larger delivery volumes and leaving room for future growth in operations without the need for additional investments. In order to rebalance the delivery operations of the library network, the authors extend a generalized assignment heuristic to handle lower and upper bound constraints on the number of stops and the load of each vehicle.

A similar application case is reported in Rienthong et al. (2011) for a mobile library service on the Isle of Wight. The problem aims to determine a set of  $m$  TSP routes for the mobile library such that every location is visited once during a recurring planning horizon of  $m$  days. The authors model the problem as an  $m$ -TSP. In order to balance the daily workload of the library, they add upper and lower bound constraints on the sum of each tour's travel and service time. An integer programming formulation of this problem is further investigated in Bektaş (2013).

Equity considerations can also be handled indirectly. Kim et al. (2006) consider a rich VRP faced by a major waste management services provider in the United States. Although total cost minimization remains the main objective, equity issues concerning route compactness and workload balance are handled indirectly through a cluster-based solution procedure. The authors find that this approach outperforms classical insertion procedures when it comes to generating well-balanced routes. Similar observations are made in Mendoza et al. (2009) in the context of periodic meter-maintenance routes for a public utility company in Colombia. In this case workload is measured by the number of maintenance visits, and equity is evaluated based on the standard deviation of the number of visits per day. The authors find that allocating visits with a cluster-first, route-second approach leads to significantly better workload balance than generating routes first and allocating them to workdays afterwards. Although the equity objective is not considered explicitly, the proposed cluster-based method produces a workload balance which is acceptable in practice.

Finally, some applications use more sophisticated functions for measuring equity. Jang et al. (2006) optimize the scheduling and routing of sales representatives of a lottery. Due to the central role of the representatives in acquiring and retaining customers, reducing overtime and increasing morale is a relevant concern. The authors measure workload equity with the mean square deviation of working hours and optimize a weighted sum of this measure combined with total travel distance. Keskinturk and Yildirim (2011) balance the tour durations for a baked goods distributor by optimizing an average relative imbalance measure as the primary objective. Kritikos and Ioannou (2010) consider the VRPTW with a balanced cargo objective, measured as the deviation from the median load. This is combined in a weighted sum with the classical VRPTW objectives of minimizing the number of used vehicles and the total distance. Lee and Ueng (1999) measure equity as the sum of the differences to the minimum tour duration.

### 3 Theoretical Analysis

In the preceding section, we observed that the VRPRB has been established in the literature as a prototypical model for an equitable VRP. The CVRP is one of the simplest VRPs, and the bi-objective model is a generalization of the weighted sum, constraint-based, and single-objective approaches. The VRPRB is thus a useful point of departure for further research.

Although many methods have already been proposed for the VRPRB, we could find no substantial discussion about the relative merits or limitations of the VRPRB model itself. Admittedly, the decision to use tour length as the equity *metric* seems to be justified in practice, since most surveyed applications also base workload either fully or at least partly on tour length. However, the reasoning for and consequences of adopting the range as the equity *function* in the VRPRB remain unclear, even though this is a defining feature of the model.

This raises the question to what extent the chosen objective affects the properties and structure of optimal VRP solutions. Some previous articles have examined this issue. Bertazzi et al. (2015) consider the worst-case trade-off between the classic min-cost and min-max optimal solutions for various VRP variants. It is shown that in the worst case, the longest route of the optimal min-cost solution is  $k$  times the longest route of the optimal min-max solution, and the min-max total distance is  $k$  times the min-cost total distance, where  $k$  is the number of vehicles. Campbell et al. (2008) study the effect of different objective functions in the context of humanitarian relief efforts, where the priority is to minimize the arrival time of aid packages. As reported by the authors, minimizing total routing cost leads to disproportionate increases in both the latest arrival time and the sum of arrival times. In contrast, minimizing the latest arrival time or the sum of arrival times leads to only moderate increases in total cost while significantly improving the service objectives. Huang et al. (2012) examine weighted sum combinations of efficiency, efficacy, and equity objectives, and how they affect tour structure and resource utilization. These studies all conclude that the choice of objective function should be well-justified because it has important theoretical and practical implications.

The articles cited above clearly demonstrate the potential drawbacks of focusing on extreme single-objective optima, but worst-case analyses offer only very limited (and possibly too conservative) guidance with respect to the properties of intermediate trade-off solutions. Yet such insight is particularly relevant in practice. Indeed, even if cost minimization remains the main objective in many applications, more equitable solutions are sought and also accepted provided that the marginal cost of equity is reasonable. In

such contexts, the primary interest lies ultimately in exploring various trade-off solutions rather than the extreme single-objective optima. A bi-objective analysis should therefore provide more generalizable conclusions.

To the best of our knowledge, Halvorsen-Weare and Savelsbergh (2016) present so far the only study examining the effects of different equity functions on VRP solutions in a bi-objective setting. The authors investigate a bi-objective MCGRP with a classical total cost objective and four possible equity objectives based on tour lengths: the range, the maximal tour, the total absolute deviation from the mean tour length, and the total absolute deviation to a desired tour length specified by a decision maker. A worst-case analysis shows that if all tours are TSP-optimal, the total distance of the range-minimizing solution is up to  $k$  times the min-cost total distance, where  $k$  is the number of vehicles. Furthermore, computational experiments suggest that different measures lead to very different total costs even at maximal equity. When examining intermediate trade-off solutions, Halvorsen-Weare and Savelsbergh (2016) illustrate that there may be large differences in equity even among solutions with the same total cost, and vice versa. Finally, the results of their computational study suggest that the size of optimal Pareto sets may vary substantially depending on the chosen equity function.

Although a number of specific equity functions have been considered in all of the cited works, there is a lack of general guidance about what *types* of functions may or may not be appropriate in a VRP context, and on what *basis* to distinguish such type categories. Given the diversity of the surveyed applications, specific function recommendations can be of only limited scope, and thus general insights are needed. To close this gap, we complement and extend the existing works by approaching the choice of equity function from an axiomatic perspective, and deriving on this basis relevant properties of the corresponding Pareto optimal solution sets.

After briefly specifying the model assumptions in Section 3.1, we recall and discuss several widely-accepted axioms for equity measures in Section 3.2, consider six classical measures in Section 3.3, and based on their properties, point out relevant implications for optimization in Section 3.4. To gauge the extent of certain implications and to assess other relevant aspects for choosing an equity measure, we report in Section 4 our observations from a numerical study on small instances solvable to optimality. Note that our conclusions are not always in agreement with previous works.

### 3.1 Model Assumptions

The aim of the VRPRB is to capture the trade-off between efficiency/cost (total workload) and equity (the fair distribution of the workload). In this analysis, we assume that workload is measured in terms of tour length, is to be minimized, and that each driver is allocated to exactly one tour (the tours in multi-trip VRPs may be suitably combined before allocating drivers). With respect to the evaluation of the two objectives, a VRP solution can be characterized as a vector  $\mathbf{x}$  of  $n$  workload allocations  $x_1, \dots, x_n$ , each corresponding to a different tour in the solution. In the VRPRB, the workload of a tour is given by its distance, so that the cost objective  $C(\mathbf{x}) = \sum_{i=1}^n x_i$ . What remains to be defined for the bi-objective problem is a function  $I(\mathbf{x})$  which expresses the inequality of a solution's workload allocation.

It is clear that issues of equitable distribution are relevant only if the resource to be distributed is finite and the number of agents is at least two. Under these conditions, equitable distribution can be seen as a multi-objective problem with  $n$  objectives, where each objective represents the allocation to a different agent. An equitable distribution should therefore be a Pareto-efficient solution to the  $n$ -objective allocation problem (Kostreva et al. 2004). However, already for small  $n$ , the proportion of non-dominated and incomparable solutions rapidly increases, so that the Pareto-dominance relation is no longer helpful for distinguishing between different allocations (Farina and Amato 2004). Similarly, if the total workload to be distributed is constant, then every possible allocation is Pareto-efficient and incomparable. Thus, the need arises for measures which reduce the dimensionality of equity and introduce a stricter ordering among allocations.

### 3.2 Desirable Properties of Inequality Measures

The choice of an inequality measure can be approached from an axiomatic perspective. The economics literature contains a large body of work on the subject of inequality measures and their desirable properties, mostly in the context of income distribution (see, e.g., Sen 1973, Allison 1978). Although there is no single accepted measure for assessing equity in general, a set of basic criteria have been identified which reasonable inequality measures should ideally satisfy. In the following we recall these criteria. We denote with  $I(\mathbf{x})$  a function (inequality measure) which assigns an index value to any allocation  $\mathbf{x}$  of  $n$  outcomes (workloads) among the set of feasible allocations  $X$ . Without loss of generality, we assume that all outcomes  $x_1, \dots, x_n$  as well as  $I(\mathbf{x})$  itself are to be minimized.

**Axiom 1.** (*Inequality Relevance*) If  $x_i = x_j$  for all outcomes  $i$  and  $j$  in  $\mathbf{x}$ , then  $I(\mathbf{x}) = 0$ . Otherwise  $I(\mathbf{x}) > 0$ .

**Axiom 2.** (*Transitivity*) Let  $I(\mathbf{x}) \geq I(\mathbf{x}')$  and  $I(\mathbf{x}') \geq I(\mathbf{x}'')$ , then  $I(\mathbf{x}) \geq I(\mathbf{x}'')$ .

The criterion of inequality relevance simply states that the measure should have a value of 0 (or any other known and fixed value) if the distribution is perfectly equitable, and be positive otherwise. Clearly, a measure should be transitive so that the ordering it produces is consistent with itself. Most inequality measures satisfy these two criteria.

**Axiom 3.** (*Scale Invariance*)  $I(\mathbf{x}) = I(\lambda\mathbf{x})$  for  $\lambda \in \mathbb{R} \setminus \{0\}$ .

**Axiom 4.** (*Translation Invariance*) Let  $\alpha \in \mathbb{R}$ , and  $\mathbf{u}$  be a unit vector of length  $n$ . Then  $I(\mathbf{x}) = I(\mathbf{x} + \alpha\mathbf{u})$ .

Scale invariance requires that the inequality index remains unchanged if all outcomes in a distribution are multiplied by a constant. This ensures that the degree of inequality of an allocation is not affected by the unit of measure (e.g. measuring distance in kilometres or miles). In contrast, translation invariance requires that the inequality index remains unchanged if a constant is added to every outcome in the distribution. Most inequality measures are either scale or translation invariant, but not both.

**Axiom 5.** (*Population Independence*)  $I(\mathbf{x}) = I(\mathbf{x} \cup \mathbf{x} \cup \dots \cup \mathbf{x})$ .

Some measures are affected by the number of outcomes in a distribution. In such cases, a direct comparison of distributions with different population sizes is not possible. One way to circumvent this limitation is to replicate each population a certain number of times such that both resulting populations are of the same size (e.g. the lowest common multiple of the original sizes). Such transformations are not necessary if the measure is population independent (Aboudi et al., 2010; Chakravarty, 2010).

**Axiom 6.** (*Anonymity*) Let  $\mathbf{x}'$  be a permutation of the elements in  $\mathbf{x}$ , then  $I(\mathbf{x}) = I(\mathbf{x}')$ .

The principle of anonymity states that the equity of an allocation should remain constant if the outcomes are permuted. Although this assumption may seem self-evident, it has far-reaching consequences for the applicability of a model. If allocation is performed over entities which are not identical (different needs, rights, preferences), then an anonymous inequality measure is not appropriate. The survey by Karsu and Morton (2015) includes a review of optimization models in which anonymity does not hold. The authors distinguish between equity (anonymity holds) and balancing (anonymity does not hold). We restrict our analysis to anonymous measures of inequality since all the surveyed VRP publications deal with this type of model.



**Axiom 7.** (*Pigou-Dalton Transfer Principle*) Let  $\mathbf{x}'$  be formed as follows:  $x'_i = x_i + \delta$ ,  $x'_j = x_j - \delta$ ,  $x'_k = x_k$ , for all  $k \notin \{i, j\}$ . If  $0 \leq \delta < x_j - x_i$ , then  $I(\mathbf{x}') \leq I(\mathbf{x})$ .

The Pigou-Dalton principle of transfers (PD) is a widely-accepted criterion for inequality measures when the aforementioned anonymity assumption holds. A transfer refers to shifting  $\delta$  units of workload from an individual  $j$  to another individual  $i$ , such that  $x'_i = x_i + \delta$  and  $x'_j = x_j - \delta$  (by definition, such transfers are *mean-preserving*). A transfer is said to be *progressive* (favouring the worse-off party) if  $0 \leq \delta \leq x_j - x_i$ , and *regressive* otherwise. The weak version of the PD principle states that if an allocation  $\mathbf{x}'$  can be reached by a finite series of only progressive transfers from an allocation  $\mathbf{x}$ , then  $I(\mathbf{x}') \leq I(\mathbf{x})$  (i.e. the new allocation cannot be less equitable). The strong version of the PD principle requires strict inequality, i.e.  $I(\mathbf{x}') < I(\mathbf{x})$ . Note that the PD principle applies only if the number and sum of outcomes of the allocations are identical.

**Axiom 8.** (*Monotonicity*) Let  $\mathbf{x}'$  be formed as follows:  $x'_i = x_i + \delta_i$  for at least one  $i$  in  $\mathbf{x}$ . If  $\delta_i \geq 0$  for all  $i$  with at least one strict inequality, then  $I(\mathbf{x}') \geq I(\mathbf{x})$ .

Recall that equitable distribution among  $n$  recipients may be considered in the form of an  $n$ -objective optimization problem. If all outcomes are to be minimized, then the inequality measure should be consistent also with the minimization of individual outcomes (Ogryczak, 2014). In other words, the inequality measure should not improve if one or more outcomes are worsened without improving any others. Formally, this means that the inequality measure should be at least weakly monotonic in all outcomes of the distribution.

### 3.3 Commonly Used Inequality Measures

Based on our survey, the review by Karsu and Morton (2015), and the broader economics literature, we have identified six classical measures of equity. In order of sophistication, they are: the worst outcome (in our context minimization of the maximal workload, which we will refer to as min-max), the lexicographic extension of min-max, the range, the mean absolute deviation, the standard deviation, and the Gini coefficient. In the following we briefly discuss these measures with respect to the axioms introduced above. A summary overview is provided in Table 2.

**Min-Max:**  $\max x_i$

Optimizing the worst outcome is likely the simplest inequality measure there is. In addition to the min-max VRP, it is also the main objective in the class of  $p$ -center facility

Property	min-max	lexicogr. min-max	range	mean abs. deviation	standard deviation	Gini coefficient
Inequality Rel.			•	•	•	•
Transitive	•	•	•	•	•	•
Scale Inv.						•
Translation Inv.			•	•	•	
Population Ind.	•	•	•	•	•	•
Anonymous	•	•	•	•	•	•
PD Consistency	weak	strong	weak	weak(+)	strong	strong
Monotonicity	weak	strong				

Table 2: Common Inequality Measures and their Axiomatic Properties

location problems as well as makespan minimization problems in scheduling (though equity concerns are technically not the focus in the latter). A justification of using min-max as an equity measure may be found in Rawls (1971), in the context of distributive justice. If an allocation problem is anonymous and impartial, then none of the entities subject to the allocation know their position in the distribution. From this point of view (termed *veil of ignorance*), individuals would rather prefer min-max distributions as they remain acceptable even in the worst case.

However, the simplicity of the min-max approach has clear disadvantages. It cannot distinguish between distributions with identical worst outcomes (min-max would be indifferent between  $[20,15,10,5]$  and  $[20,10,10,10]$ , though the second is clearly more equitable), and if the worst outcomes do differ, min-max can neither capture nor quantify the differences in equitability of the remaining parts of the distribution (e.g.  $[19,19,11,1]$  would be considered preferable to  $[20,10,10,10]$ ). Note that this remains true even if the total distributed workload is constant, as in the previous examples.

### Lexicographic Min-Max:

The lexicographic approach is a natural extension of the min-max principle. Rather than minimizing only the worst outcome, also the second-worst is minimized (subject to minimization of the first), the third-worst is minimized (subject to minimization of the previous two), and so on. The lexicographic measure satisfies the strong version of the PD principle and provides a strict preference ordering such that every distribution is assigned a unique rank in the order. This resolves the problem of distinguishing between distributions with identical worst outcomes, but does not quantify the remaining differences. Indeed, the lexicographic rank alone does not permit to make statements about *how much more* equitable one distribution is compared to another.

**Range:**  $\max x_i - \min x_i$

The range is the absolute difference between the worst and best outcomes of a distribution. Although it is the simplest measure of dispersion, it already provides significantly more information than the worst outcome alone. At the same time, it remains simple to implement and interpret, making it a popular choice in applications. Another desirable property is that the range has an unambiguous optimal value of 0 at perfect equality, unlike min-max and lexicographic min-max. However, the range does not capture the absolute levels of the outcomes (e.g. [10,9,8,7] would be preferred over [5,4,2,1], even though all workloads are lower in the latter). In addition, the range satisfies only the weak version of the PD principle, and any changes between the extremes have no effect on the index. This motivates the use of more sophisticated measures.

**Mean Absolute Deviation:**  $\frac{1}{n} \sum_{i=1}^n |x_i - \bar{x}|$

The mean absolute deviation (MAD) is defined as the mean absolute difference between each outcome and the mean outcome. Unlike the range, MAD is directly affected by every outcome in the distribution, rather than the extremes only. However, it does not satisfy the strong version of the PD principle, because transfers between outcomes on the same side of the mean do not affect this measure. Nonetheless, this is a smaller proportion of transfers than all those which do not affect the range, so in that sense, MAD is 'stronger' than the range with respect to the PD principle.

**Standard Deviation:**  $\sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2}$

The standard deviation is arguably the most well-known statistical measure of dispersion. It satisfies nearly all the aforementioned axioms. In particular, it satisfies the strong version of the PD principle, unlike three of the previous four measures. The standard deviation is translation invariant, but can easily be transformed into a scale invariant measure by dividing it by the mean of the distribution, which yields the *coefficient of variation*. Although it does not provide a strict ordering of distributions, it can be expected that in practice, distributions with identical standard deviations will be rare. The main disadvantage of the standard deviation is its computational complexity, and for decision makers it can be a less intuitive measure than simpler ones such as the range.

**Gini Coefficient:**  $\frac{1}{2n^2\bar{x}} \sum_{i=1}^n \sum_{j=1}^n |x_i - x_j|$

The Gini coefficient is one of the most widely used indices in income economics and for inequality studies in general. The standard Gini index assumes values between 0 and 1, and is easy to interpret: lower index values correspond to lower inequality. However, it

is worth noting that the bounds depend on the number of outcomes and are therefore greater/less than 0/1 for a finite set of outcomes. A translation invariant version of the index can be obtained by multiplying it with the mean of the outcomes. Like the standard deviation, the Gini coefficient satisfies the strong version of the PD principle, and similarly, the complexity of this measure is its main disadvantage in computational optimization settings.

### 3.4 Implications for Choosing an Equity Objective

Based on the foregoing analysis, we can see that none of the examined measures comply with every axiom, and no single measure dominates all others in every aspect. They can be divided into *absolute* measures (min-max and lexicographic min-max), and *relative* measures (the remaining four), depending on the monotonicity property. Both groups contain at least one measure that satisfies the strong version of the PD principle.

With respect to the PD principle, we note that it only applies when comparing solutions with the same sum of outcomes. In the VRPRB, such a case is highly unlikely if equity is based on tour lengths, because this would require two solutions with the same total distance but different tours. As a result, an inequality measure's satisfaction or not of the PD principle is not expected to have any noticeable impact on the resulting efficient sets for the standard VRPRB.

**Observation 1.** *The Pigou-Dalton transfer principle is unlikely to affect preferences between solutions if the equity metric is variable-sum, i.e.  $\sum_{i=1}^n x_i$  is not the same for all  $\mathbf{x} \in X$ .*

However, the *opposite* can be the case if equity is based on constant-sum metrics such as customer demand or service times. In such cases, every pair of feasible solutions is connected by mean-preserving transfers, and differences in equity can be captured with the PD principle whenever those transfers are strictly regressive.

On the other hand, the monotonicity property has several direct consequences for the VRPRB if workload is modelled using tour lengths, as done in most of the surveyed papers. We identify the following implications:

**Theorem 1.** *If  $I(\mathbf{x})$  is at least weakly monotonic in all  $x_i$ , then every Pareto-optimal solution to the VRPRB is composed only of TSP-optimal tours.*

*Proof.* By contradiction. Let  $\mathbf{x}$  be a Pareto-optimal solution to the VRPRB which contains at least one non-TSP-optimal tour, then there exists a solution  $\mathbf{x}'$  composed of the corresponding TSP-optima of all tours in  $\mathbf{x}$ . This implies  $C(\mathbf{x}) > C(\mathbf{x}')$  and  $I(\mathbf{x}) \geq I(\mathbf{x}')$  since  $I(\mathbf{x})$  is at least weakly monotonic in all  $x_i$ , but this contradicts the Pareto-optimality of  $\mathbf{x}$ .  $\square$

In contrast, no such guarantee can be made for non-monotonic measures. By definition, non-monotonicity implies that the deterioration of an outcome (non-TSP-optimality of a tour) can worsen, leave unchanged, but also improve the inequality index returned by a non-monotonic function. The latter case represents a potentially Pareto-optimal solution since the cost objective is strictly worsened but the equity objective is improved.

**Observation 2.** *If  $I(\mathbf{x})$  is not monotonic in all  $x_i$ , then the Pareto-optimal solutions of the VRPRB can include non-TSP-optimal tours.*

Clearly this is not a desirable property, as non-TSP-optimal tours contradict the minimization of individual workloads (Kostreva et al., 2004). More generally, non-monotonic objectives can be improved even in the extreme case when *all* workloads are strictly increased. Ogryczak (2000) consider this issue in the context of facility location models, underlining that the direct optimization of non-monotonic measures may lead to locating the facilities as far away as possible, providing almost perfectly equal *lack* of service. In a VRP setting, this corresponds to breaking the TSP-optimality of tours until all are nearly equally long. In the context of the bi-objective VRPRB and contrary to intuition, an explicit total cost objective does not resolve this issue, because an improvement in even just one objective (in this case equity) is already a sufficient condition for Pareto-optimality. We introduce the concept of **inconsistency** to distinguish solutions of this type.

**Definition 1. Inconsistency:** *A solution  $\mathbf{x}$  is inconsistent if there exists another solution  $\mathbf{x}' \in X$  such that  $C(\mathbf{x}) > C(\mathbf{x}')$  and  $x_i \geq x'_i$  for all  $i = 1 \dots n$ , but  $I(\mathbf{x}) < I(\mathbf{x}')$ .*

It is noteworthy that lower bound constraints on workload (as proposed in some papers) can directly lead to inconsistency, regardless of the equity function. Perhaps more surprisingly, even TSP-optimal solutions can be inconsistent. A different, possibly very-sub-optimal allocation of customers to tours can also increase all tour lengths. This leads to the following observation:

**Observation 3.** *TSP-optimality is not a sufficient condition for avoiding inconsistency.*

The question thus arises how monotonicity of the equity function relates to the inconsistency of the generated Pareto-optimal solutions.

**Theorem 2.** *If  $I(\mathbf{x})$  is at least weakly monotonic in all  $x_i$ , then the Pareto-optimal solutions of the VRPRB cannot be inconsistent.*

*Proof.* By contradiction. Let  $\mathbf{x}$  be an inconsistent Pareto-optimal solution to the VRPRB. According to Definition 1, there must exist another solution  $\mathbf{x}' \in X$  such that  $C(\mathbf{x}) > C(\mathbf{x}')$  and  $x_i \geq x'_i$  for all  $i = 1 \dots n$ , but  $I(\mathbf{x}) < I(\mathbf{x}')$ . If  $I(\mathbf{x})$  is at least weakly monotonic in all  $x_i$  and  $x_i \geq x'_i$  for all  $i = 1 \dots n$ , then this implies  $I(\mathbf{x}) \geq I(\mathbf{x}')$ . This contradicts the previous statement and Definition 1.  $\square$

**Observation 4.** *If  $I(\mathbf{x})$  is not monotonic in all  $x_i$ , then the Pareto-optimal solutions of the VRPRB can be inconsistent.*

Combining the results of Theorems 1 and 2, it follows that monotonic measures of equity guarantee both TSP-optimality and consistency of all generated Pareto-optimal solutions. Neither of these properties can be guaranteed for non-monotonic measures without additional restrictions. With non-monotonic inequality measures, it is therefore *possible* that some Pareto-optimal VRPRB solutions will be non-TSP-optimal and inconsistent. However, it remains unclear to what *extent* inconsistency and non-TSP-optimality are a relevant concern in practice for non-monotonic measures.

There are also a number of other aspects that are relevant for the choice of an inequality measure, but which are not directly implied by the measure's axiomatic properties. A practical issue may be the extent to which different inequality measures generate the same Pareto-optimal solutions. If there is considerable agreement between some measures, then one may prefer the simplest measure for reasons of computational complexity and ease of understanding. Likewise, if there is a noticeable difference in the average cardinality of the efficient sets, then this has practical implications (the preference for smaller or larger fronts will also depend on the application). Finally, we are also interested in the marginal cost of equity, i.e. the extent to which cost minimization and equity are conflicting objectives.

In the next section, we provide some insight into these aspects by conducting a computational study on a set of instances solvable to optimality.

## 4 Numerical Analysis

In this section, we assess numerically how the choice of equity function affects the resulting Pareto sets and the structure of Pareto-optimal VRPRB solutions between the single-objective optima. Our experiments are based on a set of small CVRP instances solvable to optimality for all of the examined measures.

We derive our test instances from the benchmark instance CMT5, which contains all the customer coordinates of the instances CMT1 to CMT10 of Christophides et al. (1979). To the best of our knowledge, in these instances the customers are distributed randomly and without any other special structure. We use the first  $n$  customers to generate a subset of instances by varying the number of vehicles  $v$  between 2 and 5, and setting the capacity  $q$  to either  $\lceil \frac{n}{v} \rceil$  (constrained) or  $\lceil \frac{n}{v} \rceil + 1$  (less constrained). Unit capacity is used for all customers. The next subset of instances is generated using the next  $n$  customers of CMT5, and so on. We generated the optimal Pareto sets by computing all feasible tours and then examining all feasible combinations of tours to derive all feasible solutions. Our study is based on instances with  $n = 14$ , as these were the largest for which Pareto-optimal sets for all parameter combinations could be enumerated within a reasonable time. We base our study on a total of 60 instances falling into 10 disjoint customer subsets. The complete set of instances used and all detailed solutions can be found online at the web address listed at the end of this paper.

### 4.1 Initial Observations

Previous studies have noted that cost-optimal VRP solutions tend to be poorly balanced (Jozefowiez et al., 2009; Reiter and Gutjahr, 2012; Bertazzi et al., 2015). We confirm those observations and find that on average, the longest tour is approximately twice as long as the shortest. This motivates the search for alternative trade-off solutions.

**Marginal Cost of Equity.** We observe that the marginal trade-off between cost and equity is generally very favourable: by selecting the solution with second-best cost, the difference between the longest and shortest tours can already be reduced by around 40% on average, while total cost increases by only around 2%. This suggests that cost and equity are not strongly conflicting objectives, and attractive compromise solutions can be found in practice.

Indeed, total cost remains the primary objective in most applications, and more equitable trade-off solutions are relevant only if their cost is within a certain gap of the

optimal cost (Reiter and Gutjahr 2012; Halvorsen-Weare and Savelsbergh 2016). We observe that a multi-objective approach yields a reasonable number of such realistic trade-off solutions: on average, 37% of efficient solutions were found to have total costs within 10% of the cost optimum (and 52% within a 15% cost gap).

**Agreement Between Equity Functions.** With respect to the actual solutions found, we observe that few are unique to any measure, but also few are common to all. Figure 1 reports the percentage of all solutions common to a given number of measures. It is interesting to note that only a small subset of solutions (around 14% or fewer in our study) is identified by *all* examined measures. Likewise, around 14% of all solutions are unique to only a single inequality measure, and around 10% to only two measures. Choosing which measure to use is thus an important decision.

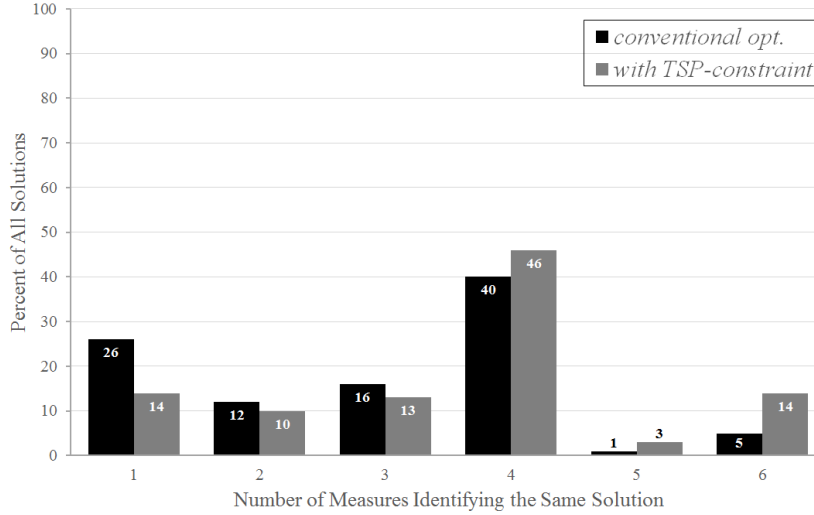


Figure 1: Solutions Identified by Multiple Inequality Measures

From initial tests we observe that a conventional solution of the instances produces, at least for non-monotonic equity functions, optimal Pareto sets which contain solutions with non-TSP-optimal tours. Jozefowiez et al. (2002) point out that tours may be artificially lengthened to distort the numerical value of the range measure. Within a heuristic framework for the VRPRB, they propose to apply a 2-opt local search to each solution prior to examining Pareto efficiency. In order to assess the extent of this phenomenon and the impact of adding extra constraints to the model, we solve all instances twice: first conventionally, and then with the constraint that all tours must be TSP-optimal. In the following, we report our observations separately for each case.



## 4.2 Conventional Optimization

We first solve each VRPRB instance to optimality without placing any additional restriction on the acceptance of solutions. In Table 3, we report for each equity function the average total size of the optimal Pareto sets, the average number of TSP-optimal solutions in those sets, and the average number of consistent solutions (see Definition 1), in absolute terms and relative to the average total. In order to give a more detailed impression of the differences between individual equity functions, we present in Table 4 the complete Pareto fronts of all examined equity functions for a representative instance. Table 4 lists for each solution the total cost, the solution’s tour lengths in decreasing order, for each examined equity function the solution’s corresponding inequality value, whether the solution is TSP-optimal, and whether the solution is consistent (white circle) or not (black circle) in the corresponding equity function’s optimal front. A dash denotes that the solution is not Pareto-optimal for the respective equity function and therefore not found in that function’s optimal Pareto set.

Comparing the two absolute measures, we can see that the lexicographic min-max yields a noticeable number of additional trade-off solutions compared to the standard min-max function (solutions 4, 6, 7, 8). It is likely that this difference will grow with instance size. In theory, the lexicographic objective should also yield more equitable solutions in those cases when there exist multiple min-max optima for the same total cost. However, in practice this special case will be rare if equity is based on tour length, because two solutions with equal length are uncommon when floating point distances are used. In our study we did not encounter such an instance.

Looking at the four relative measures, we observe that their fronts are noticeably larger than those generated with absolute measures. Although some solutions are also shared with both absolute measures (solutions 1, 2, 13, 15, 24), the Pareto sets found with relative measures are not strictly supersets of those found with absolute measures. Some solutions (e.g. 9, 12, and 23) are found only with the absolute measures, while

	min-max		lexicogr. min-max		range		mean abs. dev.		standard deviation		Gini coefficient	
	abs.	%	abs.	%	abs.	%	abs.	%	abs.	%	abs.	%
<b>Total</b>	3.53	-	4.23	-	30.22	-	43.18	-	39.37	-	38.97	-
<b>TSP-optimal</b>	3.53	100%	4.23	100%	5.58	18%	6.62	15%	6.20	16%	6.03	15%
<b>Consistent</b>	3.53	100%	4.23	100%	4.32	14%	4.55	11%	4.52	11%	4.65	12%

Table 3: Average Number of Pareto Efficient, TSP-optimal, and Consistent Solutions per Equity Function, Conventional Optimization

Nr.	Total Cost	Tour Lengths					Equity Function					TSP	Inconsistency					
		Tour 1	Tour 2	Tour 3	Tour 4	Tour 5	minmax (E1)	lex. rank (E2)	range (E3)	MAD (E4)	std. dev. (E5)		E1	E2	E3	E4	E5	E6
1	340.45	97.15	78.04	68.96	53.13	43.17	97.15	1	53.98	15.95	18.92	✓	○	○	○	○	○	○
2	341.83	83.04	78.04	68.96	68.61	43.17	83.04	2	39.87	10.08	13.74	✓	○	○	○	○	○	○
3	344.31	83.04	78.04	71.44	68.61	43.17	-	-	-	-	-						●	
4	345.43	83.04	77.81	72.80	68.61	43.17	-	3	-	-	-	✓		○			○	
5	346.41	83.04	78.04	73.20	68.96	43.17	-	-	-	-	-						●	
6	347.61	83.04	75.52	72.80	68.61	47.64	-	4	35.40	9.12	11.91	✓		○	○	○	○	○
7	350.44	83.04	75.48	72.80	68.61	50.50	-	5	32.54	8.42	10.86	✓		○	○	○	○	○
8	352.67	83.04	68.96	68.61	66.37	65.68	-	6	17.36	5.00	6.38	✓		○	○	○	○	○
9	354.26	82.81	81.28	78.04	68.96	43.17	82.81	7	-	-	-	✓	○	○				
10	355.15	83.04	71.44	68.61	66.37	65.68	-	-	-	4.97	6.33	✓				●	●	
11	355.89	83.04	69.63	68.96	68.61	65.65	-	-	-	4.75	6.09	✓		○		○	○	○
12	356.07	81.28	78.04	74.67	68.96	53.13	81.28	8	-	-	-	✓	○	○				
13	357.45	78.04	74.67	68.96	68.61	67.16	78.04	9	10.88	3.89	4.16	✓	○	○	○	○	○	○
14	359.93	78.04	74.67	71.44	68.61	67.16	-	-	-	3.50	3.97		○		●	●	●	●
15	361.05	77.81	74.67	72.80	68.61	67.16	77.81	10	10.65	3.46	3.90	✓	○	○	○	○	○	○
16	362.47	78.04	74.67	72.19	68.96	68.61	-	-	9.44	3.09	3.56	✓			●	●	●	●
17	364.23	78.04	74.67	72.19	70.36	68.96	-	-	9.09	2.81	3.23				●	●	●	●
18	364.95	78.04	74.67	72.19	71.44	68.61	-	-	-	2.69	3.18				●	●	●	●
19	365.64	77.81	74.67	73.20	72.80	67.16	-	-	-	2.52	-				●	●	●	●
20	366.08	77.81	74.67	72.80	72.19	68.61	-	-	-	2.42	3.02	✓			●	●	●	●
21	366.71	78.04	74.67	72.19	71.44	70.36	-	-	7.68	2.41	2.75				●	●	●	●
22	367.83	77.81	74.67	72.80	72.19	70.36	-	-	7.45	2.14	2.53				●	●	●	●
23	368.96	77.40	75.48	74.67	72.80	68.61	77.40	11	-	-	-	✓	○	○				
24	369.76	75.48	74.67	74.60	72.80	72.21	75.48	12	3.27	1.16	1.24	✓	○	○	○	○	○	○
25	372.01	75.48	74.67	74.60	74.46	72.80	-	-	2.69	0.64	0.88				●	●	●	●
26	405.72	81.90	81.45	81.32	81.28	79.79	-	-	2.11	0.54	0.71				●	●	●	●
27	407.69	81.90	81.75	81.45	81.32	81.28	-	-	0.62	0.23	0.25				●	●	●	●
28	492.32	98.74	98.68	98.53	98.32	98.05	-	-	-	0.22	-					●	●	●

Table 4: Pareto Optimal Fronts per Equity Function for a Representative Instance (#30), Conventional Optimization

others (e.g. 11 and 16) are found only with the relative measures. Comparing the relative measures themselves, the differences become smaller. Although in this particular instance all the solutions found by the range measure are also found by the remaining three, in general none of the relative measures is subsumed by another. We find that all the relative measures can typically be reduced to nearly 0.

With respect to the TSP-optimality of the identified Pareto-optimal solutions, one can see that TSP-optimal solutions are present at both the lower cost and lower inequality ends of all fronts, while non-TSP-optimal solutions are mainly encountered at lower inequality. Some TSP-optimal solutions (e.g. 11, 16, and 20) are identified only with relative measures, others (e.g. 9, 12, and 23) only with absolute measures.

A closer examination of the Pareto fronts reveals that some Pareto-optimal solutions are inconsistent with the minimization of individual tours (solution 28 is a particularly striking example). It is difficult to argue that inconsistent solutions could offer any practical value since their total cost is higher *and* every tour is longer (i.e. there is no real trade-off). We observe that inconsistency is mainly (though not exclusively) encountered at the low inequality end of the fronts. Most of these inconsistent solutions are not TSP-optimal, and their total solution cost rapidly increases. Solution 3 is a typical example: one of the tours of solution 2 is lengthened while the rest remain unchanged. Comparing all the equity functions, one observes that inconsistent solutions are captured only with the relative inequality measures (in line with Theorem 2 and Observation 4). Nonetheless, some TSP-optimal and consistent solutions (e.g. solution 11) are found only with relative measures. As pointed out in Observation 3, there exist solutions which are TSP-optimal but still inconsistent (e.g. solutions 16 and 20).

In order to show how these two aspects impact the structure of VRPRB solutions in the decision space, we visualize in Figure 2 two Pareto sets (min-max and range) and their corresponding solutions for another representative instance. In this example, the Pareto set obtained with the min-max measure is a subset of the Pareto set using the range measure. Solutions 1, 4, 5, and 9 are common to both Pareto fronts. Starting from the optimal cost solution, solution 2 is obtained by moving a customer from the medium-length tour to the shortest one. This improves the range but not the min-max, and therefore this solution is dominated in the min-max set. However, the lexicographic min-max would capture also solution 2. One can see that solution 3 is then obtained by increasing the length of the shortest tour. The resulting solution is no longer TSP-optimal, and is not consistent if solution 2 is also identified. Solutions 4 and 5 reassign the customers between the longest and shortest tours, yielding two TSP-optimal and

consistent solutions. Solution 6 is a typical example of how non-monotonic measures like the range may be artificially improved by breaking the TSP-optimality of a previous solution (in this case of solution 5). This behaviour is apparent again with solutions 8, 9, 10, and especially 11, the solution with minimal range. Visual inspection reveals that, in the absence of further constraints, solutions with the latter characteristics are not likely to find acceptance in practical settings.

Returning to the summary statistics in Table 3, clear distinctions in behaviour can be observed between the absolute and the relative measures. There is a marked difference in the average cardinality of the optimal Pareto sets. The relative measures capture around 10 times as many Pareto-efficient solutions as the absolute ones. The resulting Pareto fronts are unexpectedly large (around 30 to 40 solutions) considering the size of the instances. However, we also observe that only a fairly small percentage (around 15%) of the efficient solutions in these sets are TSP-optimal (in contrast to 100% with the absolute measures). If only the TSP-optimal solutions are considered, then the differences in average cardinality become smaller (ranging between 3.53 and 6.20 solutions on average), but the relative measures still consistently capture a greater number of solutions. At the same time, inconsistent solutions account for around 85% to 90% of the solutions in the Pareto sets obtained with relative measures.

As mentioned earlier, the issue of non-TSP-optimal tours in the efficient sets of the VRPRB has been noted previously by Jozefowiez et al. (2002), who propose to apply a 2-opt local search to new candidate solutions prior to examining Pareto efficiency. Given the observations made above, this suggestion appears to be well-motivated. In the following section, we therefore add an even stronger TSP-optimality constraint to the model, resolve the instances, and compare how this change impacts optimal VRPRB solution sets.

### 4.3 Optimization with TSP Constraints

We recall that the monotonic (absolute) measures (min-max and lexicographic min-max) guarantee TSP-optimality and consistency of all Pareto-efficient VRPRB solutions, as outlined in Section 3.4. Adding a TSP-optimality constraint to the model does not have any additional impact. We therefore focus our analysis in this section on the non-monotonic (relative) measures. We note that with the TSP-optimality constraint we were able to enumerate instances with up to 20 customers, but for reasons of consistency we report here the results on the same 14-customer instances used in the previous section. The results and conclusions are qualitatively the same for the larger instances.

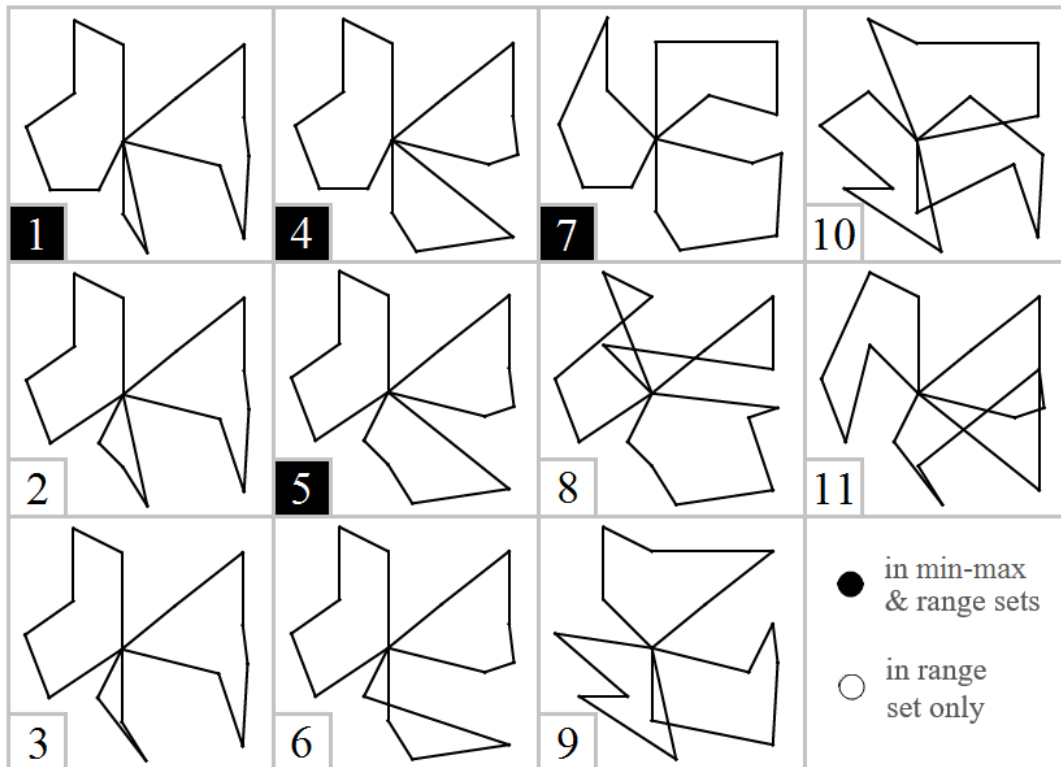
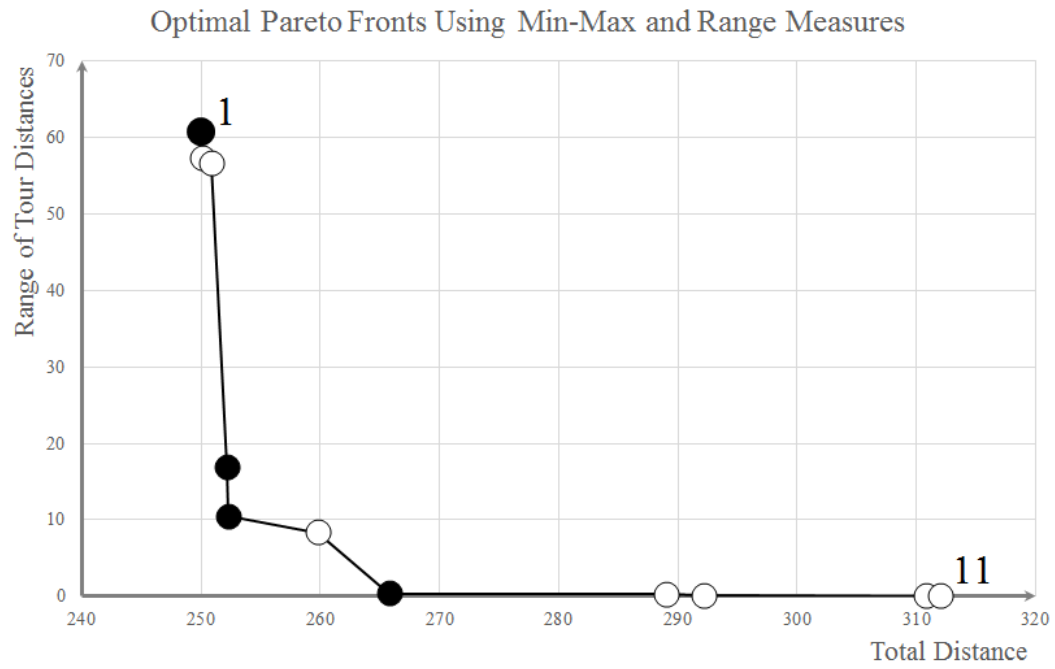


Figure 2: Visualization of VRPRB Pareto Sets and Corresponding Solutions for a Sample Instance (#46)

	range		mean abs. dev.		standard deviation		Gini coefficient	
	abs.	%	abs.	%	abs.	%	abs.	%
<b>Total</b>	14.87	-	15.85	-	16.02	-	16.45	-
<b>New</b>	9.28	62%	9.26	58%	9.86	61%	10.46	63%
<b>Consistent</b>	4.50	30%	4.52	28%	4.75	29%	4.81	29%

Table 5: Average Number of Pareto Efficient, Previously Unidentified TSP-optimal, and Consistent Solutions Found with Non-Monotonic Equity Functions, with TSP-Optimality Constraint

Given the large share of inconsistent and non-TSP-optimal solutions in the previous experiment, the question arises whether and to what extent these kinds of solutions dominate solutions that would otherwise be consistent and TSP-optimal. Table 5 presents summary statistics for the extended model. We report again the average cardinality of optimal Pareto sets. In order to more accurately assess the impact of adding the TSP-constraint, we report the absolute and relative number of *previously unidentified* (marked 'new') TSP-optimal solutions and consistent solutions. The TSP-optimal solutions identified without the TSP-constraint remain Pareto-efficient also in the extended model. To better facilitate comparisons, Table 6 lists the updated Pareto sets for the sample instance from the previous section.

We can see that with the TSP-constraint added, the average cardinality of fronts is approximately halved, but significantly more TSP-optimal solutions are identified in total. For all of the examined non-monotonic measures, approximately 60% of each new Pareto set consists of previously unidentified TSP-optimal solutions. For some instances, this figure is as high as 90%. This observation suggests that in general, a large percentage of potentially interesting solutions is dominated by non-TSP-optimal solutions in the conventional model.

**Observation 5.** *Inconsistent solutions can dominate TSP-optimal solutions which would otherwise be Pareto-optimal.*

However, a closer analysis of the new Pareto sets reveals that most of the solutions are still inconsistent. For all the examined non-monotonic measures, around 70% of the solutions identified with the extended model are inconsistent. In fact, it follows from Observation 3 that:

**Observation 6.** *Tour length minimization procedures cannot eliminate inconsistent solutions from Pareto-optimal VRPRB solution sets.*

Nr.	Tour Lengths							Equity Function					TSP	Inconsistency					
	Total Cost	Tour 1	Tour 2	Tour 3	Tour 4	Tour 5	minmax (E1)	lex. rank (E2)	range (E3)	MAD (E4)	std. dev. (E5)	Gini (E6)		E1	E2	E3	E4	E5	E6
1	340.45	97.15	78.04	68.96	53.13	43.17	97.15	1	53.98	15.95	18.92	0.16	✓	○	○	○	○	○	○
2	341.83	83.04	78.04	68.96	68.61	43.17	83.04	2	39.87	10.08	13.74	0.10	✓	○	○	○	○	○	○
3	345.43	83.04	77.81	72.80	68.61	43.17	-	3	-	-	-	0.10	✓	○	○	○	○	○	○
4	347.61	83.04	75.52	72.80	68.61	47.64	-	4	35.40	9.12	11.91	0.09	✓	○	○	○	○	○	○
5	350.44	83.04	75.48	72.80	68.61	50.50	-	5	32.54	8.42	10.86	0.08	✓	○	○	○	○	○	○
6	352.67	83.04	68.96	68.61	66.37	65.68	-	6	17.36	5.00	6.38	0.04	✓	○	○	○	○	○	○
7	354.26	82.81	81.28	78.04	68.96	43.17	82.81	7	-	-	-	-	✓	○	○	○	○	○	○
8	355.89	83.04	69.63	68.96	68.61	65.65	-	-	-	4.75	6.09	0.04	✓	○	○	○	○	○	○
9	356.07	81.28	78.04	74.67	68.96	53.13	81.28	8	-	-	-	-	✓	○	○	○	○	○	○
10	357.45	78.04	74.67	68.96	68.61	67.16	78.04	9	10.88	3.89	4.16	0.03	✓	○	○	○	○	○	○
11	361.05	77.81	74.67	72.80	68.61	67.16	77.81	10	10.65	3.46	3.90	0.03	✓	○	○	○	○	○	○
12	362.47	78.04	74.67	72.19	68.96	68.61	-	-	9.44	3.09	3.56	0.03	✓	○	○	●	●	●	●
13	366.08	77.81	74.67	72.80	72.19	68.61	-	-	-	2.42	3.02	0.02	✓	○	○	○	●	●	●
14	368.96	77.40	75.48	74.67	72.80	68.61	77.40	11	-	-	-	-	✓	○	○	○	○	○	○
15	369.76	75.48	74.67	74.60	72.80	72.21	75.48	12	3.27	1.16	1.24	0.01	✓	○	○	○	○	○	○
16	412.41	84.39	82.81	82.34	81.60	81.28	-	-	3.12	0.90	1.10	0.01	✓	○	○	●	●	●	●
17	431.98	88.07	86.66	86.49	86.17	84.58	-	-	-	0.81	-	0.01	✓	○	○	●	●	●	●
18	467.86	95.93	93.57	93.16	92.87	92.33	-	-	-	-	-	0.01	✓	○	○	○	○	○	○
19	494.01	100.83	99.48	98.19	98.02	97.50	-	-	-	-	-	0.01	✓	○	○	○	○	○	○
20	496.57	100.17	100.07	99.19	99.12	98.02	-	-	2.15	0.65	0.78	0.00	✓	○	○	●	●	●	●
21	497.99	100.82	100.17	99.19	99.12	98.69	-	-	2.13	-	-	-	✓	○	○	●	●	●	●

Table 6: Pareto Optimal Fronts per Equity Function for a Representative Instance (#30), with TSP-Optimality Constraint

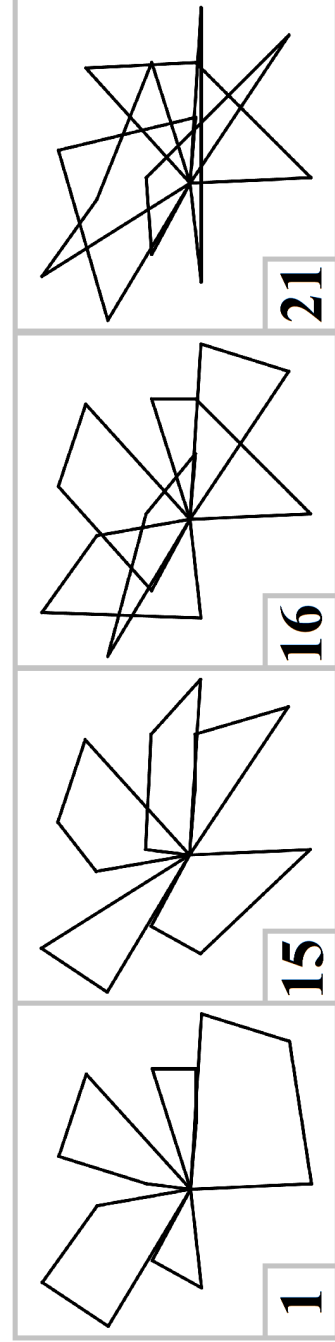


Figure 3: Visualization of Selected Solutions from Table 6

Although inconsistent solutions may still appear anywhere in the Pareto set, we observe that they are most common at the high cost/low inequality area immediately after the lexicographically minimal solution. This is not a coincidence. For the case of two vehicles (outcomes), it can be shown that all solutions with a cost higher than the lexicographic minimum are inconsistent (see Appendix 1). With a greater number of vehicles (outcomes), there is a possibility for some consistent solutions to exist beyond the lexicographic minimum. Based on our numerical study using up to five vehicles, we observe that such solutions are very rare, but do exist.

We visualize in Figure 6 some of the solutions from Table 6: the cost-optimal solution, the lexicographic minimum, the first inconsistent solution after the lexicographic minimum, and the solution with lowest inequality among all relative measures. It is difficult to imagine that a decision maker would implement solutions like solution 21, even if equity were the primary optimization objective. Numerical as well as visual inspection reveals that the equity of such solutions is artificial.

After extending the model with a TSP-constraint, all tours are TSP-optimal, but relative inequality measures are still distorted by manipulating the customer allocation sub-problem. As pointed out in Section 3.4, this is simply another way to increase the length of tours, which in turn can improve the inequality index returned by non-monotonic measures. As noted above, adding any form of tour length minimization procedure cannot solve the underlying problem caused by non-monotonicity of the equity function. With respect to the *extent* of this problem, the observations made in our numerical study strongly suggest that it is not merely a special case, but rather a common feature of Pareto-optimal solutions identified by non-monotonic measures.

We close this section by analysing the degree to which the Pareto sets using the different measures overlap. Figure 4 shows for each pair of measures the relative number of solutions unique or common to each inequality measure. Comparing the Pareto sets of the two absolute measures, we see that min-max can identify a little over 80% of the solutions found using the lexicographic measure. Similarly, we observe a high degree of overlap among relative measures, from around 50% between the range and the mean absolute deviation, to over 80% between the standard deviation and Gini coefficient. However, only around 10% of Pareto optimal solutions are common to both an absolute and a relative measure. Although the relative measures identify a much larger share of unique solutions, we have shown above that most of them are inconsistent. The degree of overlap increases for all pairs when using the TSP-constrained model.



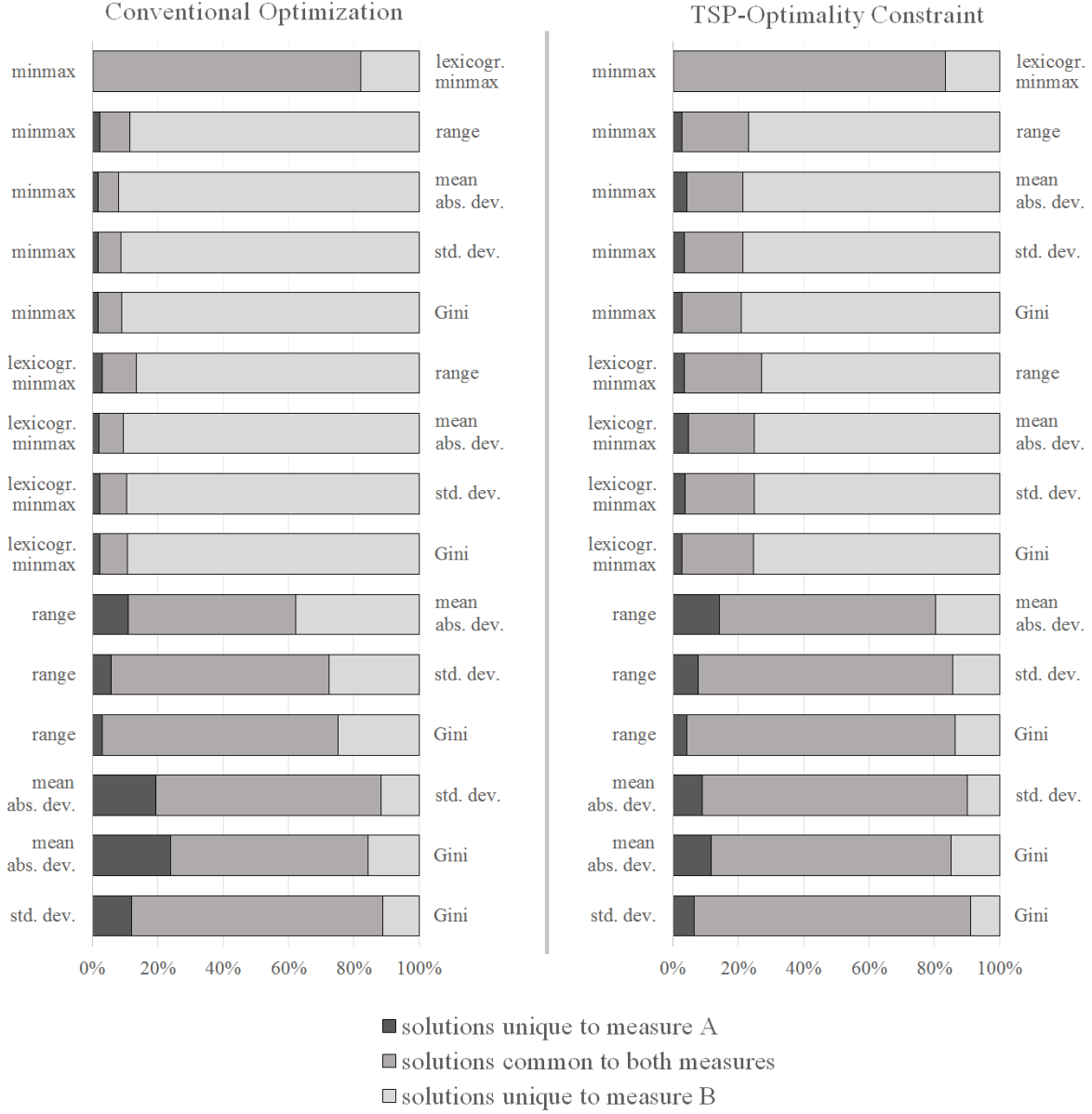


Figure 4: Overlap of Pareto Sets of Different Inequality Measures

## 5 Conclusions and Perspectives

Classifying the existing literature as a point of departure, it is clear that issues surrounding equity in logistics are gaining increasing attention from researchers and practitioners alike. The various models and methodologies proposed by researchers show that equity considerations pose interesting theoretical challenges, and the relatively large number of application-oriented papers confirm that equity concerns are also relevant in practice. Yet

despite this broad consensus about the importance and relevance of equity considerations, there has been limited discussion about just how equity should be measured and assessed in a logistics context.

We began our analysis by considering six commonly applied measures of equity and comparing them. We determined that no measure satisfies every desirable axiom and none are strictly better than others in all relevant aspects. We pointed out the implications of a measure satisfying or not the Pigou-Dalton transfer principle and the monotonicity axiom. In particular, if workload is based on tour length then non-monotonic (relative) measures of inequality can generate Pareto sets with two undesirable paradoxes: Pareto-optimal solutions may contain **non-TSP-optimal** tours, and they may also be **inconsistent** with the minimization of individual workloads, i.e. they may consist entirely of tours whose workloads are all equal to or longer than those in other efficient solutions.

In order to estimate the extent to which these issues may arise in practice and to assess other relevant aspects not directly implied by axiomatic properties (e.g. estimating the marginal cost of equity), we performed a numerical study on small instances solvable to optimality for all examined inequality measures. We confirmed the observations made in previous studies that cost-optimal VRP solutions are usually quite poorly balanced: on average, the longest tour is found to be about twice as long as the shortest. However, the marginal cost to improve equity is usually very low: on average, the second-best-cost solution reduces the gap between longest and shortest tours by around 40%, while increasing cost by only around 2%. With a view toward practical applications in which the cost criterion is likely to remain the primary objective, nearly 40% of the identified efficient solutions in our study have costs less than 10% above the optimal cost. These observations further motivate the multi-objective approach to equity in VRPs, as well as the search for an appropriate equity measure.

Our study reveals that a conventional optimization of VRPRB instances results in a very high degree of non-TSP-optimality and inconsistency in the Pareto sets identified with non-monotonic measures. On average, less than 20% of Pareto-optimal solutions were TSP-optimal, and less than 15% were consistent. The literature on the VRPRB proposes to extend the model with tour length minimization procedures (e.g. 2-opt) to filter out such solutions during optimization. Unfortunately, this does not attack the underlying cause (non-monotonicity of the equity measure), and therefore cannot entirely mitigate the mentioned effects. Nonetheless, we extended our model with a TSP-constraint in order to assess the extent to which such approaches may reduce non-TSP-optimality and inconsistency in practice. We observed that around 50% of the Pareto sets from

the extended model consist of TSP-optimal solutions which would otherwise have been dominated in the conventional approach. However, fewer than 5% of the new solutions are consistent. We therefore conclude that tour length minimization procedures are neither sufficient in theory nor in practice to overcome the problems caused by the non-monotonicity of relative inequality measures.

We thus arrive at the counter-intuitive conclusion that more sophisticated equity measures do not necessarily result in more reasonable trade-off solutions when combined with a min-sum cost objective. Although measures such as the range or the standard deviation are undoubtedly more informative when summarizing and comparing two workload allocations, they lead to some paradoxes when optimized directly. This reduces the credibility of such models, which can be especially damaging in applications. In light of these results, we conclude that the choice of the range measure for the VRPRB warrants careful re-assessment, particularly if the VRPRB is to be used as a prototypical template for equitable multi-objective VRP models.

Based on our study, monotonic measures of inequality such min-max or its lexicographic extension are more appropriate for numerical optimization when workload is based on tour length. When it comes to the choice between these two measures, the lexicographic min-max does yield an appreciable number (some 20% more on average) of unique trade-off solutions that are otherwise dominated using the standard min-max measure. These additional solutions tend to be in the low cost side of the Pareto sets, which is likely to be more interesting in practice. The disadvantages are that lexicographic optimization is more complicated to implement, and also to interpret since it is not immediately clear how the resulting fronts can be presented. Given that the size of VRP Pareto sets tends to increase exponentially, the standard min-max measure may be a more practical choice that still identifies a reasonable absolute number of trade-off solutions.

There remain a number of promising avenues for further research on equity issues in logistics. First, there is a distinct lack of models and methods for the more general case when the assumption of anonymity/impartiality does not hold. A practically-relevant example is the distinction between full-time and part-time employees, who certainly have different levels of utility or disutility for the same absolute workload. Designing adequate models for this generalization (not necessarily restricted to logistics applications) would be a significant contribution to the equity literature, upon which the development of efficient solution methods should subsequently be based.

Second, it is worthwhile to extend existing models and methods to the multi-period case. In practice, workload considerations are commonly based on a rolling horizon

perspective, especially in the case of employees' working hours. In this context, cost-minimizing solutions might appear inequitable in individual periods. Yet by allocating workloads over the horizon, it could be possible to attain an equitable overall workload distribution without significantly worsening the cost minimization objective.

Third, we have observed that some equitable VRP solutions lack credibility. Visual inspection reveals that tour workloads can be improved by simple relocations or exchanges of customers, even if individual tours are TSP-optimal. It is clear that trade-off solutions will usually not be optimal for any single objective, but if those objectives can be improved by simple visual inspection, then such solutions are not likely to be convincing in practice. The addition of local optimality constraints with neighbourhoods such as relocate or exchange is one possible way to ensure that trade-off solutions are credible without negatively impacting (heuristic) search methods. We are not aware of any studies examining the presence (or lack) of local optima in optimal Pareto sets of VRPs.

Finally, the theoretical literature that we have surveyed focuses almost exclusively on equity in terms of tour length, but the papers reporting on applications include also cases in which other factors are equally or more relevant. In the interest of generalizing existing theory and methodology, further research should place greater emphasis on broader definitions of workload. For example, in the small package delivery sector, the number of stops is the primary determinant of workload (see also Gulczynski et al. 2011 and Groër et al. 2009). Lin and Kwok (2006) and Liu et al. (2006) describe applications in telecommunications and groceries delivery, in which load/demand is at least as important as tour length. For equity metrics whose sum is constant for all solutions (number of stops, demand, service time), monotonicity of the equity function is no longer a decisive property since every increase/decrease in one outcome must be accompanied by the reverse in another. This in turn implies that the PD transfer principle becomes more important, because every pair of solutions is then connected by mean-preserving transfers. It is therefore likely that for constant-sum metrics, measures such as the range will generate richer Pareto sets without the shortcomings identified in the present study. A comparative study of equity measures for this class of metrics is therefore another promising research subject.

**Companion Web Page** The instances generated for this study, as well as detailed results for all instances, can be found at: *URL to be added*.

# Appendix

**Theorem 3.** *Let  $\mathbf{x}$  be the lexicographically minimal solution to an instance of the VRPRB. For  $n = 2$ , any solution  $\mathbf{y} \neq \mathbf{x}$  with  $C(\mathbf{y}) > C(\mathbf{x})$  is either inconsistent or dominated by  $\mathbf{x}$ .*

*Proof.* For  $n = 2$ ,  $\mathbf{x} = (x_1, x_2)$ , and  $\mathbf{y} = (y_1, y_2)$ . Let the outcomes be sorted in non-increasing order. If  $\mathbf{x}$  is lexicographically minimal, then  $x_1 \leq y_1$ . One can distinguish the following two cases with respect to the relationship between  $x_2$  and  $y_2$ :

- If  $x_1 \leq y_1$  and  $x_2 \leq y_2$ , then  $\mathbf{y}$  is inconsistent according to Definition 1.
- If  $x_1 \leq y_1$  and  $x_2 > y_2$ , then  $I(\mathbf{y}) \geq I(\mathbf{x})$  for all equity measures examined in this paper. Since  $C(\mathbf{y}) > C(\mathbf{x})$  and  $I(\mathbf{y}) \geq I(\mathbf{x})$ ,  $\mathbf{y}$  is dominated by  $\mathbf{x}$  and cannot be Pareto-optimal.

□

# References

- Aboudi, R., Thon, D., and Wallace, S. W. (2010). Inequality comparisons when the populations differ in size. *The Journal of Economic Inequality*, 8(1):47–70.
- Allison, P. D. (1978). Measures of inequality. *American Sociological Review*, pages 865–880.
- Applegate, D., Cook, W., Dash, S., and Rohe, A. (2002). Solution of a min-max vehicle routing problem. *INFORMS Journal on Computing*, 14(2):132–143.
- Apte, U. M. and Mason, F. M. (2006). Analysis and improvement of delivery operations at the san francisco public library. *Journal of Operations Management*, 24(4):325–346.
- Baños, R., Ortega, J., Gil, C., Márquez, A. L., and De Toro, F. (2013). A hybrid meta-heuristic for multi-objective vehicle routing problems with time windows. *Computers & Industrial Engineering*, 65(2):286–296.
- Bektaş, T. (2013). Balancing tour durations in routing a vehicle fleet. In *Computational Intelligence In Production And Logistics Systems (CIPLS), 2013 IEEE Workshop on*, pages 9–16. IEEE.
- Bertazzi, L., Golden, B., and Wang, X. (2015). Min–max vs. min–sum vehicle routing: A worst-case analysis. *European Journal of Operational Research*, 240(2):372–381.

- Blakeley, F., Argüello, B., Cao, B., Hall, W., and Knolmayer, J. (2003). Optimizing periodic maintenance operations for Schindler Elevator Corporation. *Interfaces*, 33(1):67–79.
- Borgulya, I. (2008). An algorithm for the capacitated vehicle routing problem with route balancing. *Central European Journal of Operations Research*, 16(4):331–343.
- Bowerman, R., Hall, B., and Calamai, P. (1995). A multi-objective optimization approach to urban school bus routing: Formulation and solution method. *Transportation Research Part A: Policy and Practice*, 29(2):107–123.
- Campbell, A. M., Vandenbussche, D., and Hermann, W. (2008). Routing for relief efforts. *Transportation Science*, 42(2):127–145.
- Carlsson, J., Ge, D., Subramaniam, A., Wu, A., and Ye, Y. (2009). Solving min-max multi-depot vehicle routing problem. *Lectures on Global Optimization. Fields Institute Communications*, 55:31–46.
- Carlsson, J. G., Carlsson, E., and Devulapalli, R. (2013). Balancing workloads of service vehicles over a geographic territory. In *Intelligent Robots and Systems (IROS), 2013 IEEE/RSJ International Conference on*, pages 209–216. IEEE.
- Chakravarty, S. R. (2010). *Inequality, Polarization and Poverty: Advances in Distributional Analysis*, volume 6. Springer Science & Business Media.
- Christophides, N., Mingozzi, A., and Toth, P. (1979). The vehicle routing problem. In Christophides, N., Mingozzi, A., Toth, P., and Sandi, C., editors, *Combinatorial Optimization*, chapter 11, pages 315–338. John Wiley, Chichester.
- Corberán, A., Fernández, E., Laguna, M., Marti, R., et al. (2002). Heuristic solutions to the problem of routing school buses with multiple objectives. *Journal of the Operational Research Society*, 53(4):427–435.
- de Armas, J., Melián-Batista, B., Moreno-Pérez, J. A., and Brito, J. (2015). GVNS for a real-world rich vehicle routing problem with time windows. *Engineering Applications of Artificial Intelligence*, 42:45–56.
- de Freitas Aquino, R. and Arroyo, J. E. C. (2014). A hybrid multi-objective iterated local search heuristic for vehicle routing problem with time windows. In *Hybrid Intelligent Systems (HIS), 2014 14th International Conference on*, pages 117–122. IEEE.

- Farina, M. and Amato, P. (2004). A fuzzy definition of "optimality" for many-criteria optimization problems. *Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on*, 34(3):315–326.
- Golden, B. L., Laporte, G., and Taillard, É. D. (1997). An adaptive memory heuristic for a class of vehicle routing problems with minmax objective. *Computers & Operations Research*, 24(5):445–452.
- Goodson, J. C. (2014). Election day routing of rapid response attorneys. *INFOR: Information Systems and Operational Research*, 52(1):1–9.
- Groër, C., Golden, B., and Wasil, E. (2009). The balanced billing cycle vehicle routing problem. *Networks*, 54(4):243.
- Gulczynski, D., Golden, B., and Wasil, E. (2011). The period vehicle routing problem: New heuristics and real-world variants. *Transportation Research Part E: Logistics and Transportation Review*, 47(5):648–668.
- Halvorsen-Weare, E. E. and Savelsbergh, M. W. (2016). The bi-objective mixed capacitated general routing problem with different route balance criteria. *European Journal of Operational Research*, 251(2):451–465.
- Huang, M., Smilowitz, K., and Balcik, B. (2012). Models for relief routing: Equity, efficiency and efficacy. *Transportation Research Part E: Logistics and Transportation Review*, 48(1):2–18.
- Jang, W., Lim, H. H., Crowe, T. J., Raskin, G., and Perkins, T. E. (2006). The Missouri Lottery optimizes its scheduling and routing to improve efficiency and balance. *Interfaces*, 36(4):302–313.
- Jozefowiez, N., Semet, F., and Talbi, E.-G. (2002). Parallel and hybrid models for multi-objective optimization: Application to the vehicle routing problem. In *Parallel Problem Solving from Nature - PPSN VII*, pages 271–280. Springer.
- Jozefowiez, N., Semet, F., and Talbi, E.-G. (2006). Enhancements of NSGA II and its application to the vehicle routing problem with route balancing. In *Artificial Evolution*, pages 131–142. Springer.
- Jozefowiez, N., Semet, F., and Talbi, E.-G. (2007). Target aiming Pareto search and its application to the vehicle routing problem with route balancing. *Journal of Heuristics*, 13(5):455–469.

- Jozefowiez, N., Semet, F., and Talbi, E.-G. (2008). Multi-objective vehicle routing problems. *European Journal of Operational Research*, 189(2):293–309.
- Jozefowiez, N., Semet, F., and Talbi, E.-G. (2009). An evolutionary algorithm for the vehicle routing problem with route balancing. *European Journal of Operational Research*, 195(3):761–769.
- Karsu, Ö. and Morton, A. (2015). Inequity averse optimization in operational research. *European Journal of Operational Research*, 245(2):343–359.
- Keskindurk, T. and Yildirim, M. B. (2011). A genetic algorithm metaheuristic for bakery distribution vehicle routing problem with load balancing. In *Innovations in Intelligent Systems and Applications (INISTA), 2011 International Symposium on*, pages 287–291. IEEE.
- Kim, B.-I., Kim, S., and Sahoo, S. (2006). Waste collection vehicle routing problem with time windows. *Computers & Operations Research*, 33(12):3624–3642.
- Kostreva, M. M., Ogryczak, W., and Wierzbicki, A. (2004). Equitable aggregations and multiple criteria analysis. *European Journal of Operational Research*, 158(2):362–377.
- Kovacs, A. A., Golden, B. L., Hartl, R. F., and Parragh, S. N. (2014). Vehicle routing problems in which consistency considerations are important: A survey. *Networks*, 64(3):192–213.
- Kritikos, M. N. and Ioannou, G. (2010). The balanced cargo vehicle routing problem with time windows. *International Journal of Production Economics*, 123(1):42–51.
- Lacomme, P., Prins, C., Prodhon, C., and Ren, L. (2015). A multi-start split based path relinking (MSSPR) approach for the vehicle routing problem with route balancing. *Engineering Applications of Artificial Intelligence*, 38:237–251.
- Lacomme, P., Prins, C., and Sevaux, M. (2006). A genetic algorithm for a bi-objective capacitated arc routing problem. *Computers & Operations Research*, 33(12):3473–3493.
- Lee, T.-R. and Ueng, J.-H. (1999). A study of vehicle routing problems with load-balancing. *International Journal of Physical Distribution & Logistics Management*, 29(10):646–657.
- Lin, C. and Kwok, R. (2006). Multi-objective metaheuristics for a location-routing problem with multiple use of vehicles on real data and simulated data. *European Journal of Operational Research*, 175(3):1833–1849.



- Liu, C.-M., Chang, T.-C., and Huang, L.-F. (2006). Multi-objective heuristics for the vehicle routing problem. *International Journal of Operations Research*, 3(3):173–181.
- Liu, R., Xie, X., and Garaix, T. (2013). Weekly home health care logistics. In *Networking, Sensing and Control (ICNSC), 2013 10th IEEE International Conference on*, pages 282–287. IEEE.
- López-Sánchez, A., Hernández-Díaz, A., Vigo, D., Caballero, R., and Molina, J. (2014). A multi-start algorithm for a balanced real-world open vehicle routing problem. *European Journal of Operational Research*, 238(1):104–113.
- Mandal, S. K., Pacciarelli, D., Løkketangen, A., and Hasle, G. (2015). A memetic NSGA-II for the bi-objective mixed capacitated general routing problem. *Journal of Heuristics*, 21(3):359–390.
- Martínez-Salazar, I. A., Molina, J., Ángel-Bello, F., Gómez, T., and Caballero, R. (2014). Solving a bi-objective transportation location routing problem by metaheuristic algorithms. *European Journal of Operational Research*, 234(1):25–36.
- Mei, Y., Tang, K., and Yao, X. (2011). Decomposition-based memetic algorithm for multi-objective capacitated arc routing problem. *Evolutionary Computation, IEEE Transactions on*, 15(2):151–165.
- Melián-Batista, B., De Santiago, A., Ángel-Bello, F., and Alvarez, A. (2014). A bi-objective vehicle routing problem with time windows: a real case in Tenerife. *Applied Soft Computing*, 17:140–152.
- Mendoza, J. E., Medaglia, A. L., and Velasco, N. (2009). An evolutionary-based decision support system for vehicle routing: The case of a public utility. *Decision Support Systems*, 46(3):730–742.
- Mourgaya, M. and Vanderbeck, F. (2007). Column generation based heuristic for tactical planning in multi-period vehicle routing. *European Journal of Operational Research*, 183(3):1028–1041.
- Nace, D. and Pióro, M. (2008). Max-min fairness and its applications to routing and load-balancing in communication networks: a tutorial. *Communications Surveys & Tutorials, IEEE*, 10(4):5–17.

- Narasimha, K. V., Kivelevitch, E., Sharma, B., and Kumar, M. (2013). An ant colony optimization technique for solving min–max multi-depot vehicle routing problem. *Swarm and Evolutionary Computation*, 13:63–73.
- Ogryczak, W. (2000). Inequality measures and equitable approaches to location problems. *European Journal of Operational Research*, 122(2):374–391.
- Ogryczak, W. (2014). Fair optimization–methodological foundations of fairness in network resource allocation. In *Computer Software and Applications Conference Workshops (COMPSACW), 2014 IEEE 38th International*, pages 43–48. IEEE.
- Oyola, J. and Løkketangen, A. (2014). GRASP-ASP: An algorithm for the CVRP with route balancing. *Journal of Heuristics*, 20(4):361–382.
- Pacheco, J. and Martí, R. (2006). Tabu search for a multi-objective routing problem. *Journal of the Operational Research Society*, 57(1):29–37.
- Pasia, J. M., Doerner, K. F., Hartl, R. F., and Reimann, M. (2007a). A population-based local search for solving a bi-objective vehicle routing problem. In *Evolutionary Computation in Combinatorial Optimization*, pages 166–175. Springer.
- Pasia, J. M., Doerner, K. F., Hartl, R. F., and Reimann, M. (2007b). Solving a bi-objective vehicle routing problem by pareto-ant colony optimization. In *Engineering Stochastic Local Search Algorithms. Designing, Implementing and Analyzing Effective Heuristics*, pages 187–191. Springer.
- Rawls, J. (1971). *A theory of justice*. Harvard University Press.
- Reiter, P. and Gutjahr, W. J. (2012). Exact hybrid algorithms for solving a bi-objective vehicle routing problem. *Central European Journal of Operations Research*, 20(1):19–43.
- Ribeiro, R. and Lourenço, H. R. D. (2001). A multi-objective model for a multi-period distribution management problem. *SSRN Working Paper Series*.
- Rienthong, T., Walker, A., and Bektaş, T. (2011). Look, here comes the library van! optimising the timetable of the mobile library service on the isle of wight. *OR Insight*, 24(1):49–62.
- Saliba, S. (2006). Heuristics for the lexicographic max-ordering vehicle routing problem. *Central European Journal of Operations Research*, 14(3):313–336.

- Sarpong, B. M., Artigues, C., and Jozefowicz, N. (2013). Column generation for bi-objective vehicle routing problems with a min-max objective. In *ATMOS-13th Workshop on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems-2013*, volume 33, pages 137–149.
- Schwarze, S. and Voß, S. (2013). Improved load balancing and resource utilization for the skill vehicle routing problem. *Optimization Letters*, 7(8):1805–1823.
- Sen, A. (1973). *On Economic Inequality*. Oxford University Press.
- Serna, C. R. D. and Bonrosto, J. P. (2001). Minmax vehicle routing problems: application to school transport in the province of Burgos. In *Computer-Aided Scheduling of Public Transport*, pages 297–317. Springer.
- Valle, C. A., Martinez, L. C., Da Cunha, A. S., and Mateus, G. R. (2011). Heuristic and exact algorithms for a min–max selective vehicle routing problem. *Computers & Operations Research*, 38(7):1054–1065.
- Wang, X., Golden, B., and Wasil, E. (2014). The min-max multi-depot vehicle routing problem: heuristics and computational results. *Journal of the Operational Research Society*, 66(9):1430–1441.
- Yakıcı, E. and Karasakal, O. (2013). A min–max vehicle routing problem with split delivery and heterogeneous demand. *Optimization Letters*, 7(7):1611–1625.